

Rearranging Alternating Harmonic Series

Note Title

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$$(1) \quad \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots + (-1)^{n+1} + \dots$$

Consider

$$(2) \quad 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \dots ,$$

where we take one positive term, two positive terms, one positive term, two positive terms, etc; in the order in which they appear in (1).

Prop: (2) converges

proof. Check that the $(3k+1)$ st partial sum of (2) is

$$s_{3k+1} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \dots + \frac{1}{2k+1} ,$$

$$s_{3k+4} = s_{3k+1} + \underbrace{\frac{1}{2k+3} - \frac{1}{4k+2} - \frac{1}{4k+4}}_{\Delta_R} ,$$

$$\Delta_R = \frac{2}{4k+6} - \frac{1}{4k+2} - \frac{1}{4k+4} < 0 .$$

this implies

$$(3) \quad s_1 > s_4 > s_7 > \dots$$

$$\text{Next } s_{3k+3} = s_{3k} + \underbrace{\frac{1}{2k+1} - \frac{1}{4k+2} - \frac{1}{4k+4}}_{d_R} ,$$

$$d_R = \frac{2}{4k+2} - \frac{1}{4k+2} - \frac{1}{4k+4} > 0 ,$$

$$\text{so } s_{3k+3} > s_{3k}$$

$$(4) \quad \dots s_9 > s_6 > s_3$$

Also, $s_{3k+1} > s_{3k}$. Hence

$$(5) \quad s_1 > s_4 > \dots > s_{3k+1} > s_{3k} > s_{3k-3} > \dots > s_6 > s_3.$$

The sequence $\{s_{3k}\}$ increases. The sequence $\{s_{3k+1}\}$ decreases. Both are bounded, by (5).

So the limits exist. $s_{3k+1} = s_{3k} + \frac{1}{2^{k+1}}$ implies that the limits are the same.

$s_{3k+3} < s_{3k+2} < s_{3k+1}$ implies that

$\lim_{n \rightarrow \infty} s_{3k+2}$ exists.

Hence $\lim_{k \rightarrow \infty} s_k$ exists, and (2) converges.

Now we can insert parentheses:

$$\begin{aligned} & \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \dots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} \\ &= \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots\right) \\ &= \frac{1}{2} \log 2 \end{aligned}$$